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E-mail:xblxb@sdu.edu.cn
http://lxbwk.njournal.sdu.edu.cn
Tel:+86-531-88366917

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程代展,清华大学毕业(1964—1970),中科院研究生院硕士(1978—1981),美国华盛顿大学博士(1981—1985)。现为中国科学院数学与系统科学研究院研究员(返聘),国际电气与电子工程师协会会士(IEEE Fellow),国际自动控制联合会会士(IFAC Fellow),中国自动化学会首届会士,曾任IFAC理事(2011—2014)及IEEE CSS执委(2010和2015),中国自动化学会控制理论专业委员会主任(2003—2010)。曾获国家自然科学二等奖两次(2008、2014,均为第一完成人),IFAC颁发的其旗舰杂志 *Automatica* 2008—2011最佳论文奖(为迄今唯一华人学者完成的获奖论文),中国科学院个人杰出成就奖(金质奖章)。此外,还获得省部级一等奖两次、二等奖四次、三等奖一次。出版学术论著17本,期刊论文300余篇,其他书籍3本。

基于图形的网络演化博弈的拓扑结构

程代展
(中国科学院数学与系统科学研究院,系统控制重点实验室, 北京 100190)

摘要:对基于图形的网络演化博弈,首先求出典型结点策略演化方程,进而给出将结点方程组合成网络局势演化方程的方法。利用局势演化方程,将计算逻辑动态系统不动点与极限环的公式推广用于图形的网络演化博弈。然后,介绍某玩家单独更新的局势演化方程,并依此给出网络演化博弈纯纳什均衡点计算公式。

关键词:网络演化博弈;局势演化方程;单独更新的局势(演化)方程; 纯纳什均衡; 矩阵半张量积

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Topological structure of graph-based networked evolutionary games

CHENG Dai-zhan
(Institute of Systems Science, Chinese Academy of Sciences, Beijing 100190, China)

Abstract: For a graph-based networked evolutionary game, the strategy evolutional equations for typical nodes are first calculated. A method is proposed to assemble typical node equations together to form the profile evolutional equation. The formula for calculating fixed points and limit cycles of logical networks is applicable to reveal the topological structure of networked evolutionary games, including the fixed points and limit cycles of networked evolutionary games. Next, for each player the unilateral profile updating equation is introduced. Using them, a formula for calculating pure Nash equilibrium(s) is obtained. Some numerical examples are presented.

Key words: networked evolutionary game; profile dynamic equation; unilateral profile updating equation; pure Nash equilibrium; semi-tensor product of matrices

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Biographies: CHENG Dai-zhan(1946—), Male, Professor, Research Interests: nonlinear control system theory. E-mail:dcheng@iss.ac.cn

1 Introduction

In recent years the theory of networked evolutionary games (NEGs) has attracted a large attention from social, computer, and systems and control communities [1,7,16]. One of the most widely investigated types of NEGs, called the graph-based NEGs, is described as following: A network graph (N,E) is given, where N is the set of players, and E is the set of players, who play a game G repeatedly. This pre-assigned game G is a two-player game, called the fundamental game.

The frequently used fundamental games could be prisoner's dilemma [18,21,14]; matching pennies [10-11,2]; hawk-dove game [19-20]; rock-paper-scissors [7], etc. For a pre-assigned fundamental game G and a fixed network graph (N,E), as long as the strategy updating rule is determined, the profile evolutional equation of the NEG, denoted by G^e, can be determined.

Two of the mostly concerned problems for NEG are: (ⅰ) topological structure of G^e, consisting of attractors (including fixed points and cycles), basin of attractions; (ⅱ) pure Nash equilibriums. The purpose of this paper is to solve some problems through building and analyzing the profile dynamics of NEGs. Following [2], the problems considered in this paper are these two.

Recently, the semi-tensor product (STP) of matrices has been used to investigate finite games. Some successful examples include: (ⅰ) modeling NEG by providing profile evolutional equations (PEEs) [7]; (ⅱ) proposing a potential equation to verify whether a game is potential or not [6]; (ⅲ) providing orthogonal decompositions of finite games based on potential and symmetry respectively [8,13], just to mention a few. Readers who are interested in the STP approach to finite games are referred to a recent survey paper [9].

An outline of the rest of this paper is as follows: Section 2 provides necessary preliminaries, including a brief survey of STP and the STP formulation for finite games. Section 3 proposes the typical node-based assembling technique to build the PEEs for overall NEGs. Then the attractors, including fixed points and cycles, of NEGs are investigated. In Section 4, the unilateral profile evolutional equations (UPEEs) for players are proposed and an algorithm is provided to calculate them. Using UPEEs, a formula is presented to calculate pure Nash equilibrium (s). Section 5 is a brief conclusion.

Before ending this section, a list of notations is provided:

(1) $\mathscr{M}_{m\times n}$: set of $m\times n$ dimensional real matrices.

(2) $\ltimes$: STP of matrices.

(3) $\mathrm{Col}(A)$ $(\mathrm{Row}(A))$: the set of columns (rows) of A; $\mathrm{Col}_i(A)$ $(\mathrm{Row}_i(A))$: the i-th column (row) of A.

(4) δ_k^i: The i-th column of identity matrix I_k.

(5) Δ_k: $\Delta_k=\mathrm{Col}(I_k)=\{\delta_k^i \mid i=1,2,\cdots,k\}$

(6) $A\in\mathscr{M}_{m\times n}$ is called a logical matrix, if $\mathrm{Col}(A)\subset\Delta_m$.

(7) $\mathscr{L}_{m\times n}$: The set of $m\times n$ logical matrices.

(8) $W_{[m,n]}$: swap matrix.

(9) $*$: Khatri-Rao product of matrices.

(10) $\mathscr{B}_{m\times n}$: the set of $m\times n$ Boolean matrices. $A=(a_{i,j})$ is a Boolean matrix, if $a_{i,j}\in\{0,1\}$ $\forall i,j$. Let $A,B\in\mathscr{B}_{m\times n}$. Then the logical operators on A and B is applied to each corresponding elements of the matrices, e.g., $A\wedge B=(a_{i,j}\wedge b_{i,j})$, etc.

(11) 1_n: $[\underbrace{1,1\cdots,1}_{n}]^{\mathrm{T}}$.

2 Preliminaries

2.1 Semi-tensor Product of Matrices

Since STP is a fundamental tool in this approach, this subsection gives a brief survey for STP. We refer to

[4-5] for more details. STP is a generalization of conventional matrix product, defined as follows:

Definition 2.1 *Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, $t=\mathrm{lcm}(n,p)$ be the least common multiple of n and p. Then the STP of A and B, denoted by $A \ltimes B$, is defined by*

$$A \ltimes B := (A\otimes I_{t/n})(B\otimes I_{t/P}), \tag{1}$$

where $\otimes$ is Kronecker product.

It is easy to see that STP is a generalization of the conventional matrix product. That is, when $n=p$, the STP is degenerated to the conventional matrix product, i.e., $A \ltimes B=AB$. Because of this, in most cases the symbol $\ltimes$ is omitted.

One of the most important advantages of STP is that STP keeps most important properties of conventional matrix product available, including association, distribution, etc. In the following we introduce some additional properties of STP, which will be used in the sequel.

The following proposition "swaps" a vector with a matrix:

Proposition 2.2 *Let $x \in \mathbf{R}^t$ be a column vector, and A be an arbitrary matrix. Then*

$$xA=(I_t\otimes A)x. \tag{2}$$

Define a swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ as follows:

$$W_{[m,n]} := [I_n\otimes\delta_m^1, I_n\otimes\delta_m^2, \cdots, I_n\otimes\delta_m^m] \in \mathcal{L}_{mn\times mn}. \tag{3}$$

Proposition 2.3 *Let $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$ be two column vectors. Then*

$$W_{[m,n]}xy=yx. \tag{4}$$

Throughout this paper the default matrix product is assumed to be STP, and the symbol $\ltimes$ is omitted if there is no possible confusion.

2　STP Expression of Finite Games

Definition 2.4 *A finite game is described by a triple (N,S,C), where*

(ⅰ) *$N=\{1,2,\cdots,n\}$ is the set of players;*

(ⅱ) *$S=\prod_{i=1}^n S_i$, which is a Cartesian product of S_i. $S_i=\{1,2,\cdots,k_i\}$ is the set of strategies of player i, $i=1,2,\cdots,n$.*

(ⅲ) *$C=(c_1,c_2,\cdots,c_n)$, where $c_i:S\to\mathbf{R}$ is the payoff function of player i, $i=1,2,\cdots,n$.*

The set of games described in Definition 2.4 is denoted by $\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$.

Using vector form expressions for strategies, we identify $j\in S_i$ with $\delta_{k_i}^j$. It turns out that $S_i\sim\Delta_{k_i}$, $i=1,2,\cdots,n$. Denote by $\kappa=\prod_{i=1}^n k_i$, then $c_i:\Delta_\kappa\to\mathbf{R}$, $i=1,2,\cdots,n$.

Let x_i be the vector form of player i's strategy. Set $x:=\ltimes_{i=1}^n x_i$, then $x\in\Delta_\kappa$ is a profile.

Proposition 2.5 *For each c_i there exists a row vector $V_i^c\in\mathbf{R}^\kappa$, called the structure vector of c_i, such that*

$$c_i(x)=V_i^c x,\ i=1,2,\cdots,n. \tag{5}$$

Consider an evolutionary game G^e, where $G\in\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$. The strategy evolutional equation (SEE) of each player can be constructed, according to strategy updating rule, as

$$x_i(t+1)=f_i(x_1(t),x_2(t),\cdots,x_n(t)),\ i=1,2,\cdots,n, \tag{6}$$

where $f_i:\prod_{j=1}^n \mathcal{D}_{k_j}\to\mathcal{D}_{k_i}$ If x_i is expressed in its vector form, i.e., $x_i\in\Delta_{k_i}$, $i=1,2,\cdots,n$, then (6) can be expressed into an algebraic form as

$$x_i(t+1)=L_i x(t),\ i=1,2,\cdots,n, \tag{7}$$

where $L_i\in\mathcal{L}_{k_i\times\kappa}$ is the structure matrix of f_i, and $x(t)=\ltimes_{j=1}^n x_j(t)$ is the profile at time t. Eventually combining all equations in (7) together yields the PEE of whole G^e as

$$x_i(t+1)=Lx(t), \tag{8}$$

where

$$L=L_1 * L_2 * \cdots * L_n \in \mathscr{L}_{\kappa\times\kappa}.$$

The f_i, $i=1,2,\cdots,n$ in (6) are determined by the strategy updating rule. In graph-based NEGs, the strategy updating rule is always assumed to be myopic best response arrangement (MBRA), which is described as follows[22]:

$$x_i(t+1)\in \operatorname{argmax}_{x_i\in S_i} c_i(x_i, x_{-i}(t)). \tag{9}$$

That is, choose the strategy at $t+1$ to be the best one against all other players' strategies at time t. When $\operatorname{argmax}_{x_i\in S_i} c_i(x_i, x_{-i}(t))$ is not unique, you can choose a fixed one (say, the one with smallest index, or with largest index, etc.) or choose any one with equal probability. They are called deterministic and stochastic types respectively.

3 Topology of NEGs

The purpose of this section is to reveal the topological structure of an NEG through its PEE. First, we need to establish the PEE of a given NEG.

3.1 Assembling PEE

Assume $N=\{N^1,N^2,\cdots,N^r\}$, where each N^i represents a kind of players. Let d_j^p be the degree of player p with set j, that is,

$$d_p^j=|\{(p,s)\in E \mid s\in N^j\}|.$$

Then two points p,q are of the same kind of players, denoted by $p\sim q$, if $p,q\in N^j$ and $d_p^j=d_q^j$, $j=1,2,\cdots,r$. Denote the degrees in this group by d_i^j. According to $\{d_i^j \mid i,j=1,2,\cdots,r\}$, the players are classified into r^r groups.

To get the PEE for whole NEG, we use the following algorithm.

Algorithm 3.1. • *Step* 1: *Calculate the PEE for each typical player, which has degree* d_j^i.

- *Step* 2: *Substitute the indexes in typical PEEs by the indexes of these players in network.*
- *Step* 3: *In the PEE of each player, put arguments (players) into an increasing order.*
- *Step* 4: *Convert the PEE of each player into full argument form.*
- *Step* 5: *Using Khatri-Rao product to assemble PEEs of all players together to form the PEE of overall NEG.*

To perform Step 3 we need to use the swap matrix for reordering the arguments. To perform Step 4 we need the following proposition.

First, assume $i_j\in\mathcal{N}_i$, then it is easy to find M_i such that

$$x_i(t+1)=M_i\ltimes_{i_j\in\mathcal{N}_i} x_{i_j}(t), \tag{10}$$

where $\mathcal{N}_i$ is the neighborhood of i.

Denote by $n_i=|\mathcal{N}_i|$. Assume all x_{i_j}, $j=1,2,\cdots,n_i$, are in an increasing order, i.e., $i_1<i_2<\cdots<i_{n_i}$. If the arguments are not in this order, the arguments need to be reordered first by using swap matrix.

Next, we construct a matrix Ξ_i as

$$\Xi_i=\otimes_{s=1}^n \xi_s, \tag{11}$$

where

$$\xi_s=\begin{cases}1_{k_s}^{\mathrm T}, & s\notin\mathcal{N}_i,\\ I_{k_s}, & s\in\mathcal{N}_i.\end{cases} \tag{12}$$

Then we have the following result:

Proposition 3.2　*Let ξ_s be defined by* (12). *Then*

$$x_i(t+1)=L_i \ltimes_{j=1}^{n} x_j(t),\ i=1,2,\cdots,n, \tag{13}$$

where

$$L_i=M_i\Xi_i.$$

Proof　For two column vectors x,y, it follows from definition that

$$x\ltimes y=x\otimes y.$$

We need the following formula[15]

$$(A\otimes B)(C\otimes D)=AC\otimes BD, \tag{14}$$

where in the right hand side AC and BD are conventional matrix product.

Using formula (14), a straightforward computation verifies (13).

Using proposition 3.2, we can have the strategy dynamic equations as

$$x_i(t+1)=L_i x(t),\ i=1,2,\cdots,n. \tag{15}$$

Finally, multiplying right hand side together we have [5]

$$x(t+1)=Lx(t), \tag{16}$$

where*

$$L=L_1 * L_2 * \cdots * L_n.$$

We use the following example to describe Algorithm 3.1.

Example 3.3[2]　*Consider a matching pennies NEG. Each player has $S_i=\{H,T\}$, where H: head, T: tail. There are two kinds of players: (1) conformist (C), (2) rebel (R). The payoff bi-matrices are shown in Tables 1, refTab.3.2, and 3, respectively for $C-C$, $R-R$, and $C-R$ respectively.*

Table 1　Payoff bi-matrix for C−C

$P_1\backslash P_2$	H	T
H	1, 1	−1,−1
T	−1,−1	1,1

Table 2　Payoff bi-matrix for R−R

$P_1\backslash P_2$	H	T
H	−1,−1	1,1
T	1,1	−1,−1

Table 3　Payoff bi-matrix for C−R

$P_1\backslash P_2$	H	T
H	1,−1	−1,1
T	−1,1	1,−1

As shown in Fig.1, there are 8 players. 4 players depicted by circles, are conformists, and 4 players depicted by squares, are rebels. Using MBRA, it is easy to figure out the strategy updating rule as follows: C-players follow the majority of their own neighborhoods. R-players against the majority of their own neighborhoods. When the number of heads equals the number of tails, player keeps his strategy unchanged.×stands for head and blank for tail. Then it is easy to verify that if at time $t=i$ the profile is depicted as in left-hand-side of Fig.1, then at time $t=i+1$ the profile becomes the right-hand side.

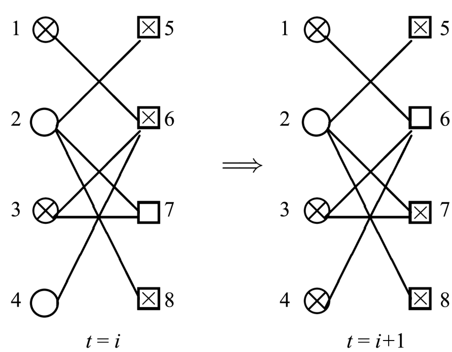

Fig.1　NEG of matching pannies

We split all players into 6 groups as C_1, C_2, C_3, R_1, R_2, R_3, according to d_1^1, d_2^1, d_3^1, d_1^2, d_2^2, and d_3^2 respectively.

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* Let $A\in\mathcal{M}_{s\times n}$, $B\in\mathcal{M}_{t\times n}$. Then the Khatri-Rao product of A and B, denoted by $A * B$, is defined by [5]

$$\mathrm{Col}_i(A * B)=\mathrm{Col}_i(A)\mathrm{Col}_i(B),\ i=1,2,\cdots,n.$$

The evolution equation for each group is easily calculated as follows.

(ⅰ) C_1: *Player x is a conformist, he has unique neighbor y. Then*

$$x(t+1)=y(t).$$

(ⅱ) C_2: *Player x is a conformist, he has two neighbors y and z. Then*

$$x(t+1)=M_2x(t)y(t)z(t),$$

and it is easy to calculate that

$$M_2=\delta_2[1,1,1,2,1,2,2,2].$$

(ⅲ) C_3: *Player x is a conformist, he has three neighbors y, z, and w. Then*

$$x(t+1)=M_3y(t)z(t)w(t),$$

and

$$M_3=\delta_2[1,1,1,2,1,2,2,2].$$

(ⅳ) R_1: *Player x is a rebel, he has unique neighbor y. Then*

$$x(t+1)=\neg\, y(t).$$

(ⅴ) R_2: *Player x is a rebel, he has two neighbors y and z. Then*

$$x(t+1)=N_2x(t)y(t)z(t),$$

and

$$N_2=\delta_2[2,2,2,1,2,1,1,1].$$

(ⅵ) R_3: *Player x is a rebel, he has three neighbors y, z, and w. Then*

$$x(t+1)=N_3y(t)z(t)w(t),$$

and

$$N_3=\delta_2[2,2,2,1,2,1,1,1].$$

Using Proposition 2, we have

$$\begin{aligned}x_1(t+1)&=x_6(t)\\&=(J_{2^5}^{\mathrm{T}}\otimes I_2\otimes J_{2^2}^{\mathrm{T}})x(t):=L_1x(t),\end{aligned}$$

where

$$L_1=J^{\mathrm{T}}\otimes I_2\otimes J_{2^2}^{\mathrm{T}}.$$

$$\begin{aligned}x_2(t+1)&=M_3x_5(t)x_7(t)x_8(t)\\&=M_3(J_{2^4}^{\mathrm{T}}\otimes I_2\otimes J_2^{\mathrm{T}}\otimes I_4)x(t):=L_2x(t),\end{aligned}$$

where

$$L_2=M_3(J_{2^4}^{\mathrm{T}}\otimes I_2\otimes J_2^{\mathrm{T}}\otimes I_4).$$

$$\begin{aligned}x_3(t+1)&=M_2x_3(t)x_6(t)x_7(t)\\&=M_2(J_4^{\mathrm{T}}\otimes I_2\otimes J_4^{\mathrm{T}}\otimes I_4\otimes J_2^{\mathrm{T}})x(t):=L_3x(t),\end{aligned}$$

where

$$L_3=M_2(J_{2^4}^{\mathrm{T}}\otimes I_2\otimes J_4^{\mathrm{T}}\otimes I_4\otimes J_2^{\mathrm{T}}).$$

$$\begin{aligned}x_4(t+1)&=x_6(t)\\&=(J_{2^5}^{\mathrm{T}}\otimes I_2\otimes J_4^{\mathrm{T}})x(t):=L_4x(t),\end{aligned}$$

where

$$L_4=J_{2^5}^{\mathrm{T}}\otimes I_2\otimes J_4^{\mathrm{T}}.$$

$$\begin{aligned}x_5(t+1)&=\neg\, x_2(t)\\&=M_\neg(J_2^{\mathrm{T}}\otimes I_2\otimes J_{2^6}^{\mathrm{T}})x(t):=L_5x(t),\end{aligned}$$

where

$$L_5=M_\neg(J_2^{\mathrm{T}}\otimes I_2\otimes J_{2^6}^{\mathrm{T}}).$$

$$\begin{aligned}x_6(t+1)&=N_3x_1(t)x_3(t)x_4(t)\\&=N_3(I_2\otimes J_2^{\mathrm{T}}\otimes I_4\otimes J_{16}^{\mathrm{T}})x(t):=L_6x(t),\end{aligned}$$

where

$$L_6=N_3(I_2\otimes J_2^{\mathrm{T}}\otimes I_4\otimes J_{16}^{\mathrm{T}}).$$

$$\begin{aligned} x_7(t+1)&=N_2x(7)x_2(t)x_3(t)\\ &=N_2W_{[4,2]}x_2(t)x_3(t)x(7)\\ &=N_2W_{[4,2]}(J_2^{\mathrm{T}}\otimes I_4\otimes J_8^{\mathrm{T}}\otimes I_2\otimes J_2^{\mathrm{T}})x(t):=L_7x(t), \end{aligned}$$

where

$$L_7=N_2W_{[4,2]}(J_2^{\mathrm{T}}\otimes I_4\otimes J_8^{\mathrm{T}}\otimes I_2\otimes J_2^{\mathrm{T}}).$$

$$\begin{aligned} x_8(t+1)&=\neg\, x_2(t)\\ &=M_{\neg}(J_2^{\mathrm{T}}\otimes I_2\otimes J_{64}^{\mathrm{T}})x(t):=L_8x(t), \end{aligned}$$

where

$$L_8=M_{\neg}(J_2^{\mathrm{T}}\otimes I_2\otimes J_{64}^{\mathrm{T}}).$$

Finally, we can calculate that the transition matrix is

$$\begin{aligned} L&=L_1*L_2*L_3*L_4*L_5*L_6*L_7*L_8\\ &=\delta_{256}[16,16,16,80,\cdots,177,241,241,241]\in\mathscr{L}_{256\times256}. \end{aligned}$$

3.2　Topology of NEGs

Consider an NEG G^e, where $G\in\mathscr{G}_{[n;k_1,k_2,\cdots,k_n]}$. Assume the PEE of G^e is

$$x(t+1)=Lx(t),\tag{17}$$

where $L\in\mathscr{L}_{\kappa\times\kappa}$. Then the number of fixed points and cycles of length s, denoted by N_s, can be calculated by the following formula, where N_1 is the number of fixed points.

Formula (18) was first obtained for Boolean networks [3], then it has been extended to k-valued logical networks [17]. Though we now are interested in a more general case as "mix-valued network", the formula is still applicable after an obvious modification.

$$\begin{cases} N_1=\mathrm{trace}(L),\\ N_s=\dfrac{\mathrm{trace}(L^s)-\sum\limits_{\mu\in\mathscr{P}(s)}\mu N_\mu}{s},\quad 2\leqslant s\leqslant\kappa. \end{cases}\tag{18}$$

Note that $\mathscr{P}(s)$ is the set of proper factors of S. For example, $\mathscr{P}(6)=\{1,2,3\}$.

Using formula (18), it is easy to verify that there is no fixed point. There are totally 4 cycles of length 4 and a cycle of length 12.

The cycle of length 12 is:

$$\begin{aligned} C^1=&\delta_{256}^{181}\to\delta_{256}^{186}\to\delta_{256}^{74}\to\delta_{256}^{69}\to\delta_{256}^{149}\to\delta_{256}^{156}\to\delta_{256}^{76}\to\delta_{256}^{71}\to\\ &\delta_{256}^{183}\to\delta_{256}^{188}\to\delta_{256}^{108}\to\delta_{256}^{101}\to\delta_{256}^{181}\to \end{aligned}$$

Converting them into component form yields

$$\begin{aligned} &(1,0,1,1,0,1,0,0)\to(1,0,1,1,1,0,0,1)\to(0,1,0,0,1,0,0,1)\to\\ &(0,1,0,0,0,1,0,0)\to(1,0,0,1,0,1,0,0)\to(1,0,0,1,1,0,1,1)\to\\ &(0,1,0,0,1,0,1,1)\to(0,1,0,0,0,1,1,0)\to(1,0,1,1,0,1,1,0)\to\\ &(1,0,1,1,1,0,1,1)\to(0,1,1,0,1,0,1,1)\to(0,1,1,0,0,1,0,0)\to\\ &(1,0,1,1,0,1,0,0)\to \end{aligned}$$

where 1: for tail, 0 for head. This result coincides with the result in [2].

The four cycles of length 4 are:

$$C_1^2:\ \delta_{256}^{1}\to\delta_{256}^{16}\to\delta_{256}^{256}\to\delta_{256}^{241}\to\delta_{256}^{1}\to$$

In component form, it is

$$\begin{aligned} &(0,0,0,0,0,0,0,0)\to(0,0,0,0,1,1,1,1)\to(1,1,1,1,1,1,1,1)\to\\ &(1,1,1,1,0,0,0,0)\to(0,0,0,0,0,0,0,0)\to \end{aligned}$$

$$C_2^2:\ \delta_{256}^{5}\to\delta_{256}^{160}\to\delta_{256}^{252}\to\delta_{256}^{97}\to\delta_{256}^{5}\to$$

In component form, *it is*

$$(0,0,0,0,0,1,0,0)\to(1,0,0,1,1,1,1,1)\to(1,1,1,1,1,0,1,1)\to$$
$$(0,1,1,0,0,0,0,0)\to(0,0,0,0,0,1,0,0)\to$$

$$C_3^2:\ (\delta_{256}^{10}\to\delta_{256}^{80}\to\delta_{256}^{247}\to\delta_{256}^{177}\to\delta_{256}^{10}\to$$

In componentwise form, *it is*

$$(0,0,0,0,1,0,0,1)\to(0,1,0,0,1,1,1,1)\to(1,1,1,1,0,1,1,0)\to$$
$$(1,0,1,1,0,0,0,0)\to(0,0,0,0,1,0,0,1)\to$$

$$C_4^2:\ \delta_{256}^{14}\to\delta_{256}^{224}\to\delta_{256}^{243}\to\delta_{256}^{33}\to\delta_{256}^{14}\to$$

In componentwise form, *it is*

$$(0,0,0,0,1,1,0,1)\to(1,1,0,1,1,1,1,1)\to(1,1,1,1,0,0,1,0)\to$$
$$(0,0,1,0,0,0,0,0)\to(0,0,0,0,1,1,0,1)\to$$

4 Pure Nash Equilibrium

Finding pure Nash equilibrium(s) for an NEG is a challenging problem, and it has been discussed a lot. We refer to [2] for more discussions. In this paper we consider how to calculate them via their SEE. Assume the SEEs of an NEG are already known as

$$x_i(t+1)=L_i x(t),\ i=1,2,\cdots,n. \tag{19}$$

Note that if MBRA is used for each player unilaterally by turn, and if $\lim_{t\to\infty}x(t)=x_0$, then x_0 must be a pure Nash equilibrium[12]. Assume player i unilaterally updates his strategy, we try to the corresponding PEE. For notational ease, we consider only k-valued games. That is, assume $|S_i|=k$, then $x_i(t)\in\Delta_k$, $i=1,2,\cdots,n$.

For the rest players, we have

$$\begin{aligned}x_{-i}(t+1)&=x_{-i}(t)=(I_{k^{i-1}}\otimes J_k^{\mathrm{T}}\otimes I_{k^{n-i}})x(t)\\&:=L_{-i}x(t)\quad i=1,2,\cdots,n.\end{aligned} \tag{20}$$

Multiplying (19) with (20) yields

$$x_i(t+1)x_{-i}(t+1)=(L_i * L_{-i})x(t). \tag{21}$$

Left multiplying both sides by $W_{[k,k^{i-1}]}$ yields

$$\begin{aligned}x(t+1)&=W_{[k,k^{i-1}]}(L_i * L_{-i})x(t)\\&:=H_i x(t).\end{aligned} \tag{22}$$

Equation (22) is called the unilateral profile evolutionary equation (UPEE) of player i. Using UPEE, the pure Nash equilibrium(s) can be calculated easily:

Proposition 4.1 *Let*

$$H:=\bigwedge_{i=1}^{n}H_i.$$

Then the number of pure Nash equilibriums of the NEG, *denoted by* n_N, *is*

$$n_N=\mathrm{trace}(H). \tag{23}$$

Moreover, *each fixed point of H is a pure Nash equilibrium.*

Proof Since for graph-based games the strategy updating rule is always assumed to be MBRA, by construct, it is clear that if $x=\delta_\kappa^s$ is the fixed point for H_i, then it means player i is not able to get better payoff by unilaterally changing his strategy. Now, if $H(s,s)=1$, then $H_i(s,s)=1$, $i=1,2,\cdots,n$, which means δ_κ^s is a common fixed point of H_i. Hence it is a pure Nash equilibrium.

Conversely, each pure Nash equilibrium can be expressed as δ_κ^s. Then it must be a fixed point for each H_i.

Example 4.2 *Consider a prisoner's dilemma*, G^e, *where the payoffs of G are described in Table* 4, *where* $T>R>P>S$.

Table 4　Payoff bi-matrix

$P_1 \backslash P_2$	C	D
C	R, R	S, T
D	T, S	P, P

Assume $x(t)=(CC)$, *then player* 1 *will take* $x(t+1)=D$. *Similarly, it is also true for other cases. Hence, the unilateral strategy updating equation for player* 1 *is*

$$x_1(t+1)=d_2[2,2,2,2]x(t):=L_1x(t). \tag{24}$$

We also have

$$\begin{aligned} x_2(t+1) &= x_{-1}(t+1)=x_{-1}(t) \\ &= (J_2^{\mathrm{T}} \otimes I_2)x(t) \\ &= \delta_2[1,2,1,2]x(t):=L_2x(t). \end{aligned} \tag{25}$$

Using (24) *and* (25), *the UPEE for player* 1 *is*

$$x(t+1)=M_1x(t), \tag{26}$$

where

$$M_1=L_1 * L_2=\delta_2[3,4,3,4].$$

Similarly, the UPEE of player 2 *is*

$$x(t+1)=M_2x(t)=\delta_4[2,2,4,4]x(t). \tag{27}$$

Then

$$H=M_1 \wedge M_2=\delta_4[0,0,0,4]. \tag{28}$$

We conclude that there is only one pure Nash equilibrium, which is $\delta_4^4 \sim (2,2)$.

Example 4.2 shows that Proposition 4.1 is also applicable to any (may be non-networked) finite game.

Example 4.3　*Recall Example* 3.3, *where the profile dynamic equations for all players have been obtained. Precisely speaking,* L_i, $i=1,2,\cdots,8$ *are known. Next, we calculate*

$$E_i=I_{2^{i-1}} \otimes J_2 \otimes 2^{8-i},\ i=1,2,\cdots,8.$$

$$M_i=W_{[2^{i-1},2]} \ltimes E_i,\ i=1,2,\cdots,8.$$

Setting

$$H=\bigcap_{i=1}^{8} M_i,$$

it is ready to verify that $\mathrm{trace}(H)=0$. *So there is no pure Nash equilibrium for matching pinnies of model depicted by Fig.*1.

Example 4.4　*Consider a public opinion game as shown in Fig.*2. *Assume* $S_i=\{P,N,U\}$, $\forall i$, *where P means positive opinion, N means negative opinion, and U is for "neutral" opinion. There are two kinds of players:* (ⅰ) *conformist* (*C*), *who follows the majority of his neighborhoods;* (ⅱ) *rebel* (*R*), *who against the majority of his neighborhoods.*

*The payoff bi-matrix for C−R is shown in Tab.*5.

Table 5　Payoff bi-matrix for C−R

$P_1(C) \backslash P_2(R)$	P	U	N
P	1, −1	0, 0	−1, 1
U	0, 0	1, 1	0, 0
N	−1, 1	0, 0	1, −1

2 *players depicted by circles, are conformists, denoted by* C_1, C_2, *and* 2 *players depicted by squares, are rebels, denoted by* D_1, D_2. *According to payoffs and using MBRA, one sees easily that* C_i *follows the majority of his own neighborhoods.* D_i *players depicted by squares, are rebels, who against the majority of their own neighborhoods. When the number of "positive" opinion equals the number of "negative" opinion, player takes a "neutral" opinion. It is easy to calculate that*

$$x_1(t+1)=Lx_3(t)x_4(t), \tag{29}$$

where

$$L=\delta_3[1,1,2,1,2,3,2,3,3].$$

$$x_2(t+1)=x_4(t). \tag{30}$$

$$x_3(t+1)=M_nLx_1(t)x_2(t). \tag{31}$$

where $M_n=\delta_3[3,2,1]$.

$$x_4(t+1)=\neg M_nx_1(t). \tag{32}$$

Then

$$\begin{aligned}x_1(t+1)&=Lx_3(t)x_4(t)\\&=L(J_9\otimes I_9)x(t):=M_1x(t).\end{aligned} \tag{33}$$

$$\begin{aligned}x_{-1}(t+1)&=(J_3^{\mathrm{T}}\otimes I_{27})x(t)\\&:=H_1x(t).\end{aligned} \tag{34}$$

$$\begin{aligned}x(t+1)&=x_1(t+1)x_{-1}(t+1)=(M_1*H_1)x(t)\\&:=\Psi_1x(t),\end{aligned} \tag{35}$$

where Ψ_1 *is calculated as*

$$\begin{aligned}\Psi_1=\delta_{81}[&1,2,30,4,32,60,34,62,63,10,11,39,13,41,69,43,71,72,\\&19,20,48,22,50,78,52,80,81,1,2,30,4,32,60,34,62,63,\\&10,11,39,13,41,69,43,71,72,19,20,48,22,50,78,52,80,81,\\&1,2,30,4,32,60,34,62,63,10,11,39,13,41,69,43,71,72,\\&19,20,48,22,50,78,52,80,81].\end{aligned}$$

Similarly, *we have*

$$x_1(t+1)=(J_{27}^{\mathrm{T}}\otimes I_3)x(t):=M_2x(t). \tag{36}$$

$$\begin{aligned}x_{-2}(t+1)&=(I_3\otimes J_3^{\mathrm{T}}\otimes I_9)x(t)\\&:=H_2x(t).\end{aligned} \tag{37}$$

$$\begin{aligned}x(t+1)&=W_{[3,3]}x_2(t+1)x_{-2}(t+1)\\&=W_{[3,3]}(M_2*H_2)x(t):=\Psi_2x(t),\end{aligned} \tag{38}$$

where Ψ_2 *is calculated as*

$$\begin{aligned}\Psi_2=\delta_{81}[&1,11,21,4,14,24,7,17,27,1,11,21,4,14,24,7,17,27,\\&1,11,21,4,14,24,7,17,27,28,38,48,31,41,51,34,44,54,\\&28,38,48,31,41,51,34,44,54,28,38,48,31,41,51,34,44,54,\\&55,65,75,58,68,78,61,71,81,55,65,75,58,68,78,61,71,81,\\&55,65,75,58,68,78,61,71,81].\end{aligned}$$

$$\begin{aligned}x_3(t+1)&=M_nLx_1(t)x_2(t)\\&=M_nL(I_9\otimes J_9^{\mathrm{T}})x(t):=M_3x(t).\end{aligned} \tag{39}$$

$$\begin{aligned}x_{-3}(t+1)&=(I_9\otimes J_3^{\mathrm{T}}\otimes I_3)x(t)\\&:=H_3x(t).\end{aligned} \tag{40}$$

$$\begin{aligned}x(t+1)&=W_{[3,9]}x_3(t+1)x_{-3}(t+1)\\&=W_{[3,9]}(M_3*H_3)x(t):=\Psi_3x(t),\end{aligned} \tag{41}$$

where Ψ_3 *is calculated as*

$$\begin{aligned}\Psi_3=\delta_{81}[&7,8,9,7,8,9,7,8,9,16,17,18,16,17,18,16,17,18,\\&22,23,24,22,23,24,22,23,24,34,35,36,34,35,36,34,35,36,\\&40,41,42,40,41,42,40,41,42,46,47,48,46,47,48,46,47,48,\\&58,59,60,58,59,60,58,59,60,64,65,66,64,65,66,64,65,66,\\&73,74,75,73,74,75,73,74,75].\end{aligned}$$

$$x_4(t+1)==\neg x_1(t)=M_n(I_3\otimes J_{27}^{\mathrm{T}})x(t)$$
$$:=M_4x(t). \quad (42)$$
$$x_{-4}(t+1)=(I_{27}\otimes J_3^{\mathrm{T}})x(t)$$
$$:=H_4x(t). \quad (43)$$
$$x(t+1)=x_{-4}(t+1)x_4(t+1)$$
$$=(H_4*M_4)x(t):=\Psi_4x(t), \quad (44)$$

where Ψ_4 *is calculated as*

$$\Psi_4=\delta_{81}[3,3,3,6,6,6,9,9,9,12,12,12,15,15,15,18,18,18,$$
$$21,21,21,24,24,24,27,27,27,29,29,29,32,32,32,35,35,35,$$
$$38,38,38,41,41,41,44,44,44,47,47,47,50,50,50,53,53,53,$$
$$55,55,55,58,58,58,61,61,61,64,64,64,67,67,67,70,70,70,$$
$$73,73,73,76,76,76,79,79,79].$$

Note that (35), (38), (41), *and* (44) *are UPEEs of players* 1,2,3, *and* 4 *respectively. Setting* $H=\Psi_1\wedge\Psi_2\wedge\Psi_3\wedge\Psi_4$, *it is easy to calculate that*

$$\mathrm{trace}(H)=1,$$

and $\mathrm{Col}_{41}(H)=\delta_{81}^{41}$, *hence the only pure Nash equilibrium is*

$$\delta_{81}^{41}\sim(\delta_3^2,\delta_3^2,\delta_3^2),$$

which corresponds to (U,U,U).

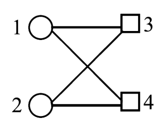

Fig.2　NEG of public opinion

5　Conclusion

Graph-based NEGs were investigated. First, using STP, the PEEs for typical nodes were constructed as bricks. Then an algorithm was proposed to assemble all “bricks” together to form the PEE of the overall NEG. Using PEE, main topological properties of NEG, including fixed points and cycles, were calculated. Second, the UPEE for individual player was proposed and algorithm was presented to calculate them for all players respectively. Using UPEE, a formula was obtained to calculate Nash equilibrium(s) of NEGs.

References:

[1] BRAMOULLE Y, KRANTON R. Games played on networks[M]//BRAMOULLE Y, GALEOTTI A, ROGERS B, et al. The Oxford Handbook of the Economics of Networks, Chapter 5. Oxford: Oxford University Press, 2016: 83-112.

[2] CAO Z G, QIN C Z, YANG X G, et al. Dynamic matching pennies on networks[J]. International Journal of Game Theory, 2019, 48(3):887-920. doi:10.1007/s00182-019-00665-5.

[3] CHENG D Z, QI H S. A linear representation of dynamics of Boolean networks[J]. IEEE Transactions on Automatic Control, 2010, 55(10):2251-2258. doi:10.1109/TAC.2010.2043294.

[4] CHENG D Z, QI H S, LI Z. Analysis and control of Boolean networks: a semi-tensor product Approach[M]. London: Springer, 2011.

[5] CHENG D Z, QI H S, ZHAO Y. An introduction to semi-tensor product of matrices and its applications[M]. Singapore: World Scientific, 2012. DOI:10.1142/8323.

[6] CHENG D Z. On finite potential games[J]. Automatica, 2014, 50(7):1793-1801. doi:10.1016/j.automatica.2014.05.005.

[7] CHENG D Z, HE F H, QI H S, et al. Modeling, analysis and control of networked evolutionary games[J]. IEEE Transactions on Automatic Control, 2015, 60(9):2402-2415. doi:10.1109/TAC.2015.2404471.

[8] CHENG D Z, LIU T, ZHANG K Z, et al. On decomposed subspaces of finite games[J]. IEEE Transactions on Automatic

Control, 2016, 61(11):3651-3656. doi:10.1109/TAC.2016.2525936.

[9] CHENG D Z, WU Y H, ZHAO G D, et al. A comprehensive survey on STP approach to finite games[EB/OL]. [2021-06-30]. J Sys Sci Compl (to appear), http://arxiv.org/abs/2106.16086.

[10] CHIAPPORI P A, LEVITT S, GROSECLOSE T. Testing mixed-strategy equilibria when players are heterogeneous: the case of penalty kicks in soccer[J]. American Economic Review, 2002, 92(4):1138-1151. doi:10.1257/00028280260344678.

[11] EASLEY D, KLEINBERG J. Networks, crowds, and markets: reasoning about a highly connected world [M]. Cambridge: Cambridge University Press, 2010. DOI:10.1017/cbo9780511761942.

[12] GUO P L, WANG Y Z, LI H T. Algebraic formulation and strategy optimization for a class of evolutionary networked games via semi-tensor product method[J]. Automatica, 2013, 49(11):3384-3389. doi:10.1016/j.automatica.2013.08.008.

[13] HAO Y Q, CHENG D Z. On skew-symmetric games[J]. Journal of the Franklin Institute, 2018, 355(6):3196-3220. doi:10.1016/j.jfranklin.2018.02.015.

[14] HILBE C, NOWAK M A, SIGMUND K. Evolution of extortion in iterated prisoner's dilemma games[J]. Proc Natl Acad Sci, 2013, 110(17):6913-6918. doi:10.1073/pnas.1214834110.

[15] HORN R A, JOHNSON C R. Matrix analysis[M]. Cambridge: Cambridge University Press, 1986.

[16] JACKSON MO, ZENOU Y. Games on networks[M]//PEYTON Y, ZAMIR S. Handbook of game Theory: Chapter 3, Vol 4. Amsterdam: Elsevier Science, 2014: 95-164.

[17] LI Z Q, CHENG D Z. Algebraic approach to dynamics of multivalued networks[J]. International Journal of Bifurcation and Chaos, 2010, 20(3):561-582. doi:10.1142/s0218127410025892.

[18] PRESS W H, DYSON F J. Iterated prisoner's dilemma contains strategies that dominate any evolutionary opponent[J]. Proc Natl Acad Sci, 2012, 109(26):10409-10413. doi:10.1073/pnas.1206569109.

[19] SMITH J M, PRICE G R. The logic of animal conflict[J]. Nature, 1973, 246(5427):15-18. doi:10.1038/246015a0.

[20] SMITH J M. Evolution and the theory of games[M]. Cambridge: Cambridge University Press, 1982.

[21] STEWART A J, PLOTKIN J B. Extortion and cooperation in the prisoner's dilemma[J]. Proc Natl Acad Sci, 2012, 109(26):10134-10135. doi:10.1073/pnas.1208087109.

[22] YOUNG H P. The evolution of conventions[J]. Econometrica, 1993, 61(1):57. doi:10.2307/2951778.

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